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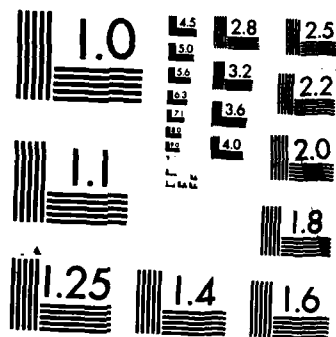
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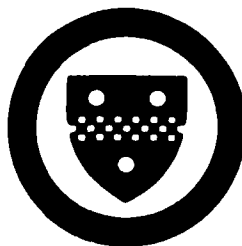
by

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CHARACTERIZATION OF DISCRETE PROBABILITY  
DISTRIBUTIONS BY PARTIAL INDEPENDENCE

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*Key Words and Phrases:* Characterization by conditional distribution; generalized Polya-Eggenberger distribution; Moran's theorem, Poisson distribution; partial independence.

ABSTRACT

→ If  $X$  and  $Y$  are random variables such that  $P(X > Y) = 1$  and the conditional distribution of  $Y$  given  $X$  is binomial, then Moran (1952) showed that  $Y$  and  $(X-Y)$  are independent if  $X$  is Poisson.

*This document* We extend Moran's result to a more general type of conditional distribution of  $Y$  given  $X$ , using only partial independence of  $Y$  and  $X-Y$ . This provides a generalization of a recent result of Janardhan and Rao (1982) on the characterization of generalized Polya-Eggenberger distribution. A variant of Moran's theorem is proved which generalizes the results of Patil and Seshadri (1964) on the characterization of the distribution of a random variable  $X$  based on some conditions on the conditional distribution of  $Y$  given  $X$  and the independence of  $Y$  and  $X-Y$ . ↗

## 1. INTRODUCTION

If  $X$  and  $Y$  are random variables such that  $P(X \geq Y) = 1$ ,  $X$  is Poisson and the conditional distribution of  $Y$  given  $X = z$  is binomial with parameters  $z$  and  $p$ , a simple calculation shows that  $X - Y$  and  $Y$  are independent and Poisson. Moran (1952) proved the following converse result. Let

$$P(X = n, Y = r) = \binom{n}{r} p_n^r (1 - p_n)^{n-r} g_n, \quad r = 0, 1, \dots, n; \quad n \geq 0 \quad (1.1)$$

where  $g_n = P(X = n)$ , with  $g_0 < 1$ , and  $p_n \in (0, 1)$  and fixed for each  $n$ . Then  $Y$  and  $(X - Y)$  are independent iff  $p_n = \text{constant}$  independent of  $n$ , and  $X$  is Poisson.

Chatterji (1963) proved the same result by replacing (1.1) by the weaker conditions

$$\begin{aligned} P(X = r, Y = r) &= p_r^r g_r \\ P(X = r+1, Y = r) &= (r+1) p_r^{r+1} (1 - p_{r+1}) g_{r+1} \\ P(X = r+2, Y = r) &= 2^{-1} (r+1) (r+2) p_{r+2}^r (1 - p_{r+2})^2 g_{r+2} \end{aligned} \quad (1.2)$$

$$r = 0, 1, 2, \dots$$

Recently, some extensions of Moran's result have been obtained by Gerber (1980) and Janardhan and Rao (1982). In this paper, we prove some general results which provide some extensions and refinements of the results of the above quoted authors. We shall refer to their work in the remarks following the main theorems.

## 2. AN EXTENSION OF MORAN'S RESULT

Let  $\{(a_n, b_n, h_n) : n = 0, 1, \dots\}$  be a sequence of real vectors such that  $a_n > 0$ ,  $b_n \geq 0$ ,  $h_n > 0$  for all  $n \geq 0$  with  $b_0, b_1$  and  $b_2 > 0$ . Define the family of conditional distributions

$$F(r|n) = c_n^{-1} h_n^r a_n^{n-r} h_n^{-(n-r)} b_{n-r} \quad (2.1)$$

$$r = 0, 1, \dots, n; \quad n \geq 0$$

where  $c_n$  are such that  $F(1|n) + \dots + F(n|n) = 1$ ,  $n \geq 0$ . We prove the following general theorem.

**Theorem 2.1.** Let  $(X, Y)$  be a 2-vector random variable with non-negative integer components, and denote  $P(X=n) = g_n$  and  $P(Y=r|X=n) = S(r|n)$ . Assume that  $P(X=0) = g_0 < 1$  and  $S(r|n) = F(r|n)$  as defined in (2.1). Then the following are equivalent:

- (i)  $P(Y=r|X=Y) = P(Y=r|X=Y+1) = P(Y=r|X=Y+2)$ ,  $r=0,1,\dots$
- (ii)  $h_n = A \alpha^n$  and  $g_n = (g_0/c_0) c_n \beta^n$ ,  $n=0,1,\dots$ , for some  $A$ ,  $\alpha$  and  $\beta > 0$ .
- (iii)  $Y$  is independent of  $X-Y$ .

**Proof.** It is easy to check (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (i). So, it is sufficient to show that (i)  $\Rightarrow$  (ii) to prove the theorem. Suppose that (i) is valid. Then

$$\frac{g_r S(r|r)}{P(X=Y)} = \frac{g_{r+1} S(r|r+1)}{P(X=Y+1)} = \frac{g_{r+2} S(r|r+2)}{P(X=Y+2)}, \quad r=0,1,\dots \quad (2.2)$$

In view of the assumptions imposed on the sequence  $(a_n, b_n, h_n)$ :  $n \geq 0$ , (2.2) implies that  $g_r > 0$  for all  $r \geq 0$ . Consequently

$$\begin{aligned} \frac{g_{r+2}}{g_{r+1}} \frac{c_{r+1}}{c_{r+2}} &= \frac{P(X=Y+1)}{P(X=Y)} \frac{h_{r+1}^{r+1} b_0}{h_{r+2}^r b_1} \\ &= \frac{P(X=Y+2)}{P(X=Y+1)} \frac{h_{r+1}^{r-1} b_1}{h_{r+2}^{r-2} b_2}, \quad r=0,1,\dots \end{aligned} \quad (2.3)$$

and hence

$$\frac{h_{r+2}}{h_{r+1}} = \left[ \frac{P^2(X=Y+1)}{P(X=Y)P(X=Y+2)} \frac{b_0 b_2}{b_1^2} \right]^{1/2} = \alpha \text{ (say), } r=0,1,\dots$$

which implies that

$$h_r = A \alpha^r, \quad r=1,2,\dots \quad (2.4)$$

for some positive constant  $A$ . Substituting (2.4) in (2.3) leads to

$$\frac{g_{r+2}}{c_{r+2}} = \beta \frac{g_{r+1}}{c_{r+1}}, \quad r = 0, 1, \dots$$

where  $\beta = b_0 A \alpha P(X=Y+1)/b_1 P(X=Y)$ . Hence, using (2.2) with  $r=0$

$$\frac{g_r}{c_r} = \frac{g_0}{c_0} \beta^r, \quad r = 1, 2, \dots$$

which completes the proof.

The following corollary is an improved version of Moran's result.

Corollary 1. Let  $(X, Y)$  be as in Theorem 2.1 with

$$S(r|n) = \binom{n}{r} p_n^r (1-p_n)^{n-r}, \quad r = 0, 1, \dots, n; \quad n \geq 0 \quad (2.5)$$

for some fixed sequence  $\{p_n; p_n \in (0, 1], n = 0, 1, \dots\}$ . Then Theorem 2.1 holds with (ii) replaced by

(ii)'  $p_n = \text{constant}$  independently of  $n$  and  $\{g_n\}$  is a Poisson distribution.

Proof. Note that (2.5) can be written as

$$S(r|n) = \frac{\binom{n}{r} \left(\frac{p_n}{1-p_n}\right)^{r/2} \left(\frac{p_n}{1-p_n}\right)^{-(n-r)/2}}{[p_n(1-p_n)]^{-n/2}}, \quad r = 0, 1, \dots, n; \quad n \geq 0.$$

This is of the form (2.1) with  $a_n = b_n = (n!)^{-1}$  and

$$h_n = \left(\frac{p_n}{1-p_n}\right)^{1/2} \quad \text{and} \quad c_n = \frac{[p_n(1-p_n)]^{-n/2}}{n!}.$$

If (i) is valid, then Theorem 2.1 implies

$$[p_n/(1-p_n)] = a\gamma^n, \quad n = 1, 2, \dots$$

for some positive constants  $a$  and  $\gamma$ . Hence

$$c_n = (\gamma^{-n/2} + a\gamma^{n/2})a^{-n/2}/n!, \quad n = 1, 2, \dots$$



which implies that, for appropriate  $B$  and  $\beta$ ,

$$g_n = B\beta^n a^{-n/2} (\gamma^{-n/2} + a\gamma^{n/2})^n / n!, \quad n = 0, 1, \dots$$

Since  $\sum_{n=0}^{\infty} g_n = 1$ , we must have  $\gamma = 1$  in which case  $\{g_n\}$  is a Poisson distribution and  $p_n$  is a constant. This completes the proof.

Remark 1. If the condition (i) in Theorem 2.1 is replaced by

$$P(Y=r) = P(Y=r | X=Y+j), \quad j = 2, 3, \dots; \quad r \geq 0 \quad (2.6)$$

then the conclusion of the theorem does not remain valid as the following example shows. Let

$$g_n = \begin{cases} e^{-\lambda} \lambda^n / n!, & n = 2, 3, \dots \\ \alpha e^{-\lambda} \lambda, & n = 1 \\ e^{-\lambda} (1 + 1 - \alpha \lambda), & n = 0 \end{cases}$$

and

$$S(r|n) = \begin{cases} \binom{n}{r} (\alpha p)^r (1 - \alpha p)^{n-r}, & r = 0, 1, \dots, n; \quad n \geq 2 \\ \binom{1}{r} p^r (1-p)^{1-r}, & r = 0, 1; \quad n = 1 \end{cases}$$

where  $0 < \alpha, p < 1$  and  $\lambda > 0$ . It is easy to see that if  $(X, Y)$  is the corresponding random vector, then it satisfies (2.6); but the distribution of  $X$  here is not of the form given in (ii) of Theorem 2.1.

Remark 2. Gerber (1980) obtained a multivariate generalization of Moran's theorem with some restrictions on the distributions of the individual random variables. He established a result similar to the equivalence of our conditions (ii) and (iii) of Theorem 2.1. The main interest of our theorem is in showing the equivalence of the condition (i) with (ii) and (iii).

We prove the following theorem which extends the results of Janardhan and Rao (1982). We need the following definitions. A non-negative integer valued random variable  $X$  is said to have

a GPED  $(h, t, c, \lambda)$ , i.e., generalized Polya-Eggenberger distribution, if

$$P(X=r) = K(h, t, c, \lambda) \frac{h(h+rt)^{(r,c)}}{(h+rt)r!} \lambda^r, \quad r=0,1,\dots \quad (2.7)$$

for some  $\lambda > 0$ ,  $h > 0$ ,  $t > 0$ ,  $(c+t) \geq 0$  such that the distribution is well defined, with  $K(h, t, c, \lambda)$  as the normalizing constant, where the notation

$$m^{(r,c)} = m(m+c)\dots(m+r-1c)$$

is used. A non-negative integer valued random variable  $X$  is said to have a GMPD  $(n, a, b, c, t)$ , i.e., generalized Markov-Polya distribution if

$$P(X=n) = \binom{n}{r} \frac{ab(a+b+nt)(a+rt)^{(r,c)}(b+n-rt)^{(n-r,c)}}{(a+b)(a+rt)(b+n-rt)(a+b+nt)^{(n,c)}} \quad (2.8)$$

$r = 0, 1, \dots, n$ , where  $a > 0$ ,  $b > 0$ ,  $t \geq 0$  such that  $(c+t) \geq 0$ . [We note that neither GPED or GMPD as defined above possesses the identifiability property in terms of its parameters. However, this does not effect our results in the present investigation. Also it is possible to define these distributions with some of the parameters as negative. We do not deal with such cases here.]

**Theorem 2.2.** Let  $\{(a_n, b_n): n=1,2,\dots\}$  be a sequence of real vectors with positive components such that  $a_n + b_n = A$  (a fixed constant) for all  $n \geq 1$ . Further let  $(X, Y)$  be a 2-vector random variable with non-negative integral components such that  $g_0 < 1$ , where  $g_n = P(X=n)$ , and the survival distribution  $S(\cdot|n)$  is GMPD  $(n, a_n, b_n, c, t)$ . Then the following conditions are equivalent:

$$(i) \quad P(Y=r|X=Y) = P(Y=r|X=Y+1); P(X-Y=r|Y=0) = P(X-Y=r|Y=1), \\ r = 0, 1, \dots \quad (2.9)$$

(ii)  $X$  has GPED  $(A, t, c, \lambda)$  for some  $\lambda > 0$  and  $a_n = \text{constant}$  independently of  $n$ .

(iii)  $Y$  and  $X-Y$  are independent.

**Proof.** It is easy to check (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (i). We shall show that (i)  $\Rightarrow$  (iii). Clearly (i) is equivalent to

$$\left. \begin{aligned} S(r|r)g_r/P(X=Y) &= S(r|r+1)g_{r+1}/P(X=Y+1) \\ S(0|r)g_r/P(Y=0) &= S(1|r+1)g_{r+1}/P(Y=1) \end{aligned} \right\} r = 0, 1, \dots \quad (2.10)$$

The fact that  $a_r, b_r > 0$  for all  $r > 1$  and  $g_0 < 1$  implies in view of (2.10) that  $g_r > 0$  for all  $r > 0$  and hence (2.10) gives

$$[S(r|r)/S(0|r)] = dS(r|r+1)/S(1|r+1), \quad r = 0, 1, 2, \dots \quad (2.11)$$

where  $d = P(Y=1)P(X=Y)/P(X=Y+1)P(Y=0)$ . Since  $S(r|n)$  is of the form (2.8) we can rewrite (2.11) as

$$\frac{a_r}{b_r} \frac{(a_r + rt + c)^{(r-1, c)}}{(b_r + rt + c)^{(r-1, c)}} = d \frac{(a_{r+1} + rt + c)^{(r-1, c)}}{(b_{r+1} + rt + c)^{(r-1, c)}}, \quad r = 1, 2, \dots \quad (2.12)$$

Suppose that  $(a_2/b_2) \neq d$ . Then (2.12) implies that  $(a_r/b_r)$  is strictly monotonic which contradicts the assumption that at least two  $a_r$ 's are equal. Then  $(a_2/b_2) = d$  and hence, using (2.12), we conclude inductively that  $a_r = \text{constant}$  independently of  $r$ . Substituting in (2.10), we get for appropriate  $B$  and  $\lambda$

$$\begin{aligned} g_{r+1} &= B \frac{S(r|r)}{S(r|r+1)} g_r \\ &= \lambda \frac{(A+rt)}{(A+r+1t)} \frac{(A+r+1t)^{(r+1, c)}}{(A+rt)^{(r, c)}} \frac{g_r}{(r+1)} \\ &= \lambda^{r+1} \frac{(A+r+1t)^{(r+1, c)}}{(A+r+1t)^{(r+1)!}} g_0, \quad r = 0, 1, 2, \dots \end{aligned}$$

Hence  $\{g_n\}$  is GPED  $(A, t, c, \lambda)$ . This completes the proof.

**Remark 3.** Theorem 2.2 is an improvement over that of Janardhan and Rao (1982) in the sense that the condition of independence is weakened by (2.9). On the other hand, the special choice that  $a_n$ 's are equal for at least two values of  $n$  puts a restriction on the sequence  $\{a_n\}$ . However, this latter condition can be avoided for the case when  $(c+t) > 0$  as follows.

If  $(a_s/b_s) > d$  for some  $s > 1$ , then (2.12) implies that  $\{a_r/b_r : r=s, s+1, \dots\}$  is increasing and tends to a limit, say  $a \leq A$ . The equation (2.12) then yields that

$$\frac{a_r}{b_r} \prod_{k=1}^{r-1} \{1 + O_k(\frac{1}{r})\} = d \prod_{k=1}^{r-1} \{1 + O'_k(\frac{1}{r})\},$$

where  $O_k$  and  $O'_k$  are smaller order functions of  $(1/r)$  uniformly for  $k = 1, 2, \dots, r-1$ . The above equation implies that  $\{a_r/b_r\}$  converges to  $d$ . However, this leads to a contradiction since the monotonic increasing nature of  $\{a_r/b_r : r \geq s\}$  implies that of  $\{(a_r/b_r) : r \geq s\}$  and hence it is impossible that

$$\lim_{r \rightarrow \infty} \frac{a_r}{b_r} = d < \frac{a_s}{b_s}.$$

Hence, we cannot have  $s$  such that  $(a_s/b_s) > d$ . By symmetry, we can also conclude that it is impossible that  $(a_s/b_s) < d$  for some  $s$ . Consequently  $(a_s/b_s) = d$  for all  $s \geq 1$ .

Remark 4. As corollaries of Theorems 2.1 and 2.2, one can obtain characterizations of many well known discrete probability distributions by special choices of  $\{(a_n, b_n), n \geq 0\}$ . For instance an improved version of Chatterjee's (1963) result is as follows:

Theorem 2.3. Let  $(X, Y)$  be such that

$$S(r|r) = h_r^r a_r b_0 / c_r,$$

$$S(r|r+1) = h_{r+1}^{r-1} a_r b_1 / c_{r+1},$$

and

$$S(r|r+2) = h_{r+1}^{r-2} a_r b_2 / c_{r+2},$$

$$r = 0, 1, 2, \dots$$

where  $a_n, b_n, c_n$  and  $h_n$  are defined in Theorem 2.2. If

$$P(Y=r|X=Y) = P(Y=r|X=Y+1) = P(Y=r|X=Y+2), \quad r = 0, 1, \dots,$$

then

$$h_n = A\alpha^n \quad \text{and} \quad g_n = (g_0/c_0)c_n\beta^n, \quad n=1,2,\dots,$$

for some constants  $A$ ,  $\alpha$  and  $\beta > 0$ .

Theorem 2.4. Let  $(X,Y)$  be such that

$$S(r|r) = \frac{a_r(A+rt)(a_r+rt)^{(r,c)}}{(a_r+rt)A(A+rt)^{(r,c)}}, \quad r=0,1,2,\dots$$

$$S(r|r+1) = (r+1) \frac{a_{r+1}b_{r+1}(A+r+1t)(a_{r+1}+rt)^{(r,c)}}{(a_{r+1}+rt)A(A+r+1t)^{(r+1,c)}}$$

$$S(0|r) = \frac{b_r(A+rt)(b_r+rt)^{(r,c)}}{(b_r+rt)A(A+rt)^{r,c}}$$

$$S(1|r+1) = (r+1) \frac{a_{r+1}b_{r+1}(A+r+1t)(b_{r+1}+rt)^{(r,c)}}{(b_{r+1}+rt)A(A+r+1t)^{(r+1,c)}}$$

where  $a_n$ ,  $b_n$  and  $A$  are as defined in Theorem 2.2. If the assertion (i) of Theorem 2.3 is valid, then  $a_n = \text{constant}$  independently of  $n$ , and  $\{g_n\}$  is GEPD  $(A,t,c,\lambda)$  for some  $\lambda > 0$ .

Remark 5. If  $b_n = a_n = 1/n!$   $n \geq 0$  and  $h_n = [p_n/(1-p_n)]^{1/2}$  for all  $n \geq 1$  in Theorem 2.3, then we obtain Chatterjee's (1963) result.

### 3. SOME VARIANTS OF MORAN'S RESULTS

Woodbury (1949) has considered a general Bernoulli scheme in which the probability of a success at a given trial depends on the number of successes in previous trials. Let  $p_r \in (0,1)$  be the probability of a success in a trial given that  $(r-1)$  of the earlier trials resulted in successes. If  $C(r|n)$  denotes the probability of  $r$  successes in  $n$  trials, then it is easy to check the following

$$C(r|n) = p_r C(r-1|n-1) + (1-p_{r+1})C(r|n-1); r=0,1,\dots,n; n \geq 1$$

with

$$C(r|0) = \delta_{0r}, \quad r = 0,1,\dots \quad (3.1)$$

where  $\delta_{ij}$  is the Kronecker delta. Many authors have studied this model. Suzuki (1980) has given a historical sketch of this model and studied in some detail the special case where  $p_r = p$  for  $r \leq m$  and  $p_r = \gamma p$  for  $r > m$ ,  $m > 1$ , where  $p$  and  $\gamma$  are real numbers in  $(0,1)$ . This model can be generalized to the case where the probability of a success depends also on the number of trials. If  $p_{r,n}$  denotes the probability in such a case, then we have

$$C(r|n) = p_{r,n} C(r-1|n-1) + (1-p_{r+1,n})C(r|n-1); r=0,1,\dots,n; n \geq 1,$$

with

$$C(r|0) = \delta_{0r}, \quad r = 0,1,\dots \quad (3.2)$$

Now, let  $Z_1, Z_2, \dots$  be a sequence of non-degenerate independent (0-1 valued) Bernoulli random variables and let  $(X,Y)$  be as defined in the previous section such that  $P(X=0) = g_0 < 1$  and

$$S(r|n) = P\left(\sum_{i=1}^n Z_i = r\right), \quad r=0,1,\dots,n \quad (3.3)$$

for all  $n$  for which  $g_n > 0$ . If  $\{p_n\}_{n=1}^{\infty}$  is the corresponding sequence of the probability of success, then it is not difficult to see that  $S(r|n)$  given in (3.3) satisfies (3.2) with  $p_{r,n} = p_n$  for all  $r=0,1,\dots,n$  and each  $n$  (i.e., the one in which the probability of success depends only on the number of trials). Kimeldorf et al. (1981) used this model and they established the following result.

**Theorem 3.1.** Let  $X, Y, Z_1, Z_2, \dots$  be as defined above such that  $p_n$  are equal for at least two values of  $n$ . Then

$$P(Y=r|X=Y) = P(Y=r|X=Y+1), \quad P(X-Y=r|Y=0) = P(X-Y=r|Y=1); \\ r = 0,1,\dots, \quad (3.4)$$

iff  $p_n = \text{constant}$  independently of  $n \geq 1$  and  $g_n$  is a Poisson distribution.

The following theorem shows that a similar result holds when the survival distribution is taken as the one resulting from Woodbury scheme for Bernoulli trials.

Theorem 3.2. Let  $(X, Y)$  be a random vector with non-negative integer-valued components such that

$$P(Y=r, X=n) = C(r|n)g_n, \quad r = 0, 1, \dots, n; \quad n \geq 0,$$

where  $\{C_n(r|n): r = 0, 1, \dots, n, n \geq 0\}$  is as defined in (3.1) and  $g_0 < 1$ . Suppose that for at least two distinct integers  $n_1$  and  $n_2$ , we have  $p_{n_1} = p_{n_2}$ . Then the following are equivalent:

- (i) Condition (3.4) is valid.
- (ii)  $p_n = \text{constant}$  independently of  $n$  and  $\{g_n\}$  is Poisson.
- (iii)  $Y$  is independent of  $X-Y$ .

Proof: It is not difficult to see (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (i) and hence it is sufficient to show that (i)  $\Rightarrow$  (ii). Suppose (i) is valid. Then we have

$$\begin{aligned} C(r|r)g_r/P(X=Y) &= C(r|r+1)g_{r+1}/P(X=Y+1) \\ \text{and} \quad r &= 0, 1, \dots \end{aligned} \quad (3.5)$$

$$C(0|r)g_r/P(Y=0) = C(1|r+1)g_{r+1}/P(Y=1).$$

From (3.1) it follows inductively that

$$\begin{aligned} C(r|r) &= \prod_{i=1}^r p_i \\ C(r|r+1) &= \prod_{i=1}^r p_i \sum_{j=1}^{r+1} (1-p_j) \\ C(0|r) &= (1-p_1)^r \\ C(1|r+1) &= p_1 \sum_{i=1}^{r+1} (1-p_1)^i (1-p_2)^{r+1-i} \end{aligned} \quad r = 1, 2, \dots$$

Hence (3.5) implies that

$$\sum_{i=1}^{r+1} (1-p_1) = c \sum_{i=1}^{r+1} [(1-p_2)/(1-p_1)]^{r+1-i} \quad r=1,2,\dots$$

for some  $c > 0$ . This leads to

$$1-p_{r+1} = c[(1-p_2)/(1-p_1)]^r, \quad r=2,3,\dots$$

Since  $p_{n_1} = p_{n_2}$  and  $n_1 \neq n_2$ , we must have  $p_2 = p_1$  and therefore  $p_r = p_1$  for all  $r = 2,3,\dots$ . Substituting in (3.5), we see that  $g_{r+1} = \lambda g_r / (r+1)$ , where  $\lambda = [P(X=Y+1)/P(X=Y)](1-p_1)$ . This yields that  $X$  has a Poisson distribution.

Theorems 3.1 and 3.2 raise the question concerning the validity of the result when the survival distributions are obtained from a generalized Woodbury's model (i.e. when the probabilities of successes are allowed to depend on both the number of successes and the order of trials). The answer to this question is in the negative. For example, if we set  $p_{r,n} = \frac{a+r-lc}{a+b+n-lc}$  in (3.2) for some  $c > 0$ , then induction argument implies that for each  $n \geq 0$ ,  $C(r,n)$  is MPED  $(n,a,b,c,0)$ . Hence Theorem 3 applies for any  $c > 0$  but  $p_{r,n}$  is not independent of either  $r$  or  $n$ .

#### 4. AN EXTENSION OF PATIL-SESHADRI RESULT

Patil and Seshadri (1964) have given characterizations of some discrete and continuous distributions based on a specific form of the conditional distribution of  $Y$  given  $X$  and independence of  $Y$  and  $X-Y$ . Recently, Panaretos (1982) extended the Patil-Seshadri result by considering only partial independence of  $Y$  and  $X-Y$ . In the following theorem, we prove the basic results of Panaretos under less restrictive conditions.

Theorem 4.1. Let  $(X,Y)$  be a random vector of non-negative real valued components such that  $P(X \geq Y) = 1$ , and there exist integers  $n_0 = 0 < n_1 < \dots < n_r$  (with  $r \geq 1$ ) such that  $P(Y \geq n_i - n_{i-1}) > 0$ ,  $i = 0,1,\dots,r$ , and every integer in  $(n_r, n_r + n_{r1})$ , where  $n_{r1}$  is the minimum of  $(n_i - n_{i-1})$  over  $1 \leq i \leq r$ , is contained in the semigroup



generated by  $\{0, 1, \dots, n_r\}$ . Also assume that a version of the conditional distributions  $\{S(m|n)\}$  is such that  $S(m|m+n_i) > 0$  for  $i = 0, 1, \dots, r$  and  $m = 0, 1, \dots, Y_R$ , the right extremity of the distribution of  $Y$ . Then the condition

$$P(Y=m|X-Y=0) = P(Y=m|X-Y=n_i), \quad m = 0, 1, \dots; i = 1, \dots, r \quad (4.1)$$

implies that

$$P(X=n) = P(X=0)c_n \lambda^n, \quad n = 1, \dots, n_r + Y_R$$

for some sequence  $\{c_n\}$  determined uniquely by  $\{S(m|n)\}$  and some positive  $\lambda$ .

Proof. The condition (4.1) implies that

$$\frac{g_m S(m|n)}{P(X=Y)} = \frac{g_{m+n_i} S(m|m+n_i)}{P(X=Y+n_i)}, \quad m = 0, 1, \dots, Y_R; i = 1, \dots, r. \quad (4.2)$$

Define for  $0 \leq n \leq n_r + Y_R$

$$u_n = \begin{cases} 1 & \text{if } g_n = 0 \\ 0 & \text{if } g_n \neq 0. \end{cases}$$

Then it follows from (4.2) that  $u_n = 0$  on  $\{n_r, n_r+1, \dots, n_r+n_{ir}\}$ , and hence  $u_n = u_0$  for all  $n$ . Since there is at least one positive  $g_n$  for  $0 \leq n \leq n_r + Y_R$ , in view of  $P(X=Y) > 0$ , we can then conclude

$$g_n > 0 \text{ for all } n \in \{0, 1, \dots, n_r + Y_R\}.$$

From (4.2), we have the following identity with both sides positive

$$\frac{g_{m+1}}{g_m} = \frac{S(m+1|m+1+n_i)}{S(m+1|m+1)} \frac{S(m|m)}{S(m|m+n_i)} \frac{g_{m+1+n_i}}{g_{m+n_i}}. \quad (4.3)$$

Define now for  $0 \leq n \leq n_r + Y_R - 1$

$$v_m = \begin{cases} 1 & \text{if } n = 0, \\ \\ \text{the } v_m \text{ corresponding to the minimum of} \\ m < n \text{ such that } (g_{m+1}/g_m) = c(m,n) (g_{n+1}/g_n) \\ \text{for some } c(m,n) \text{ determined uniquely by} \\ \{S(\cdot|\cdot)\} \text{ if such a representation is possible} \\ \text{and } n \geq 1, \\ \\ \max_{0 \leq m < n} v_m + 1 & \text{if the representation is not} \\ \text{possible and } n \geq 1. \end{cases}$$

Now, essentially the same argument as in the case of  $\{u_n\}$  implies that  $v_n = v_0$  for all  $n \in \{0, 1, \dots, n_r + Y_R - 1\}$ . Consequently, it follows that for every  $1 \leq n \leq n_r + Y_R - 1$

$$\frac{g_{n+1}}{g_n} = [c(m,n)]^{-1} \frac{g_1}{g_0}$$

which implies the required result with  $\lambda = (g_1/g_0)$  for  $2 \leq n \leq n_r + Y_R$ . The result for  $n=1$  easily follows.

Remark 6. If  $S(m|n)$  is taken to be of the form  $a_m b_{n-m}/c_n$ , then the condition on  $S(m|n)$  in the theorem reduces to  $a_m > 0$  for all  $0 \leq m \leq Y_R$  and  $b_0, b_{n_1}, \dots, b_{n_r} > 0$  with other  $b_i$ 's non-negative. In that case, clearly (4.1) implies that

$$P(X=n) = P(X=0) (c_n/c_0) \lambda^n, \quad n = 0, 1, \dots, n_r + Y_R,$$

for some  $\lambda > 0$ , where  $\{c_n\}$  is the convolution of  $\{a_n\}$  and  $\{b_n\}$ .

Remark 7. If we replace in the Theorem, the condition that every integer in  $(n_r, n_r + n_{r1})$  is contained in the semigroup generated by  $\{0, n_1, \dots, n_r\}$  by that the largest common divisor of  $n_1, \dots, n_r$ ,  $i = 1, \dots, r$ , is unity, the result does not remain valid.

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X based on some conditions on the conditional distribution of Y given X and the independence of Y and X-Y.

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